Research Statement

Jennifer Kenkel

1. Introduction

My research is focused on commutative algebra, using techniques from representation theory and analysis, and with applications to algebraic geometry. The focus of commutative algebra is on classifying the behavior of rings and modules. I work on local cohomology modules, large and abstractly defined modules that give information about rings and their ideals. I have used techniques from analysis to construct explicit examples of elements of local cohomology modules, and techniques from representation theory to calculate explicit lengths of local cohomology modules. Most of my research has focused on the class of determinantal rings. I will expand my work to more cases, and use the information gathered to develop a theory on the relationship between local cohomology modules and the structure of the underlying ring and ideal.

Homological invariants, such as Ext, Tor, and local cohomology modules are central to the focus of commutative algebra, not only because they give information about rings and the maps between them, but because they also are fascinating modules with rich structures in their own right. Local cohomology modules, in particular, are invaluable tools. The local cohomology modules associated to a ring are an indexed sequence of objects that detect both the size of the ring and aspects of the ring's behavior. Groethendieck showed that the last non-zero local cohomology module of a ring, called the top local cohomology, occurs at the dimension of the ring. In a similar vein, the first non-zero local cohomology module occurs at index equal to an invariant known as 'depth,' which measures the behavior of the ring. In algebraic geometry, local cohomology has been applied to proving Lefschetz-type theorems. In commutative algebra, it has also been used to obtain theorems on arithmetic rank, the least number of polynomials needed to define an algebraic set. One can get an upper bound on arithmetic rank by demonstrating a set of polynomials that define the algebraic set, but showing this collection of polynomials is minimal is practically impossible without the use of local cohomology modules.

Even when derived from simple rings, local cohomology modules are defined homologically, and thus difficult to work with explicitly. They are subtle and challenging objects. For example, top local cohomology modules for a ring of positive dimension never have finite length. In fact, some local cohomology modules are so large that they contain an embedding of every finite abelian group [Cha12]. In general, much of the work that has been done on local cohomology modules examines whether the module is or is not zero; however, there has been a growing trend towards studying how the modules themselves behave, despite their sometimes infinite length and intractable nature [BBL+16, DM17, DM18, DS18].

The quotient of a ring R by an ideal I has a sequence of thickenings, R/I^t. The natural map on thickenings, $R/I^{t+1} \to R/I^t$ induces a map on local cohomology modules:

$$H^k_{\mathfrak{m}}(R/I^{t+1})_j \to H^k_{\mathfrak{m}}(R/I^t)_j.$$

Bhatt et al. examined these maps as t grows arbitrarily large in [BBL+16]. The authors found that, for R/I a locally complete intersection, the induced maps on local cohomology modules are eventually isomorphisms in each graded component, when the index of the local cohomology is less than the codimension of the singular locus. However, an isomorphism is not guaranteed for the modules as a whole, as the thickening parameter t for which a degree stabilizes will depend on the degree. The next natural question, if isomorphisms in every degree cannot be guaranteed, is to consider how far these maps between local cohomology modules are from being isomorphisms. To do so, algebraists have considered the lengths of the local cohomology modules of R/I^t as t increases.

My research program is focused on two goals: explicitly constructing isomorphisms between local cohomology modules of thickenings of rings, and using representation theoretic work, specifically Schur functors, to measure lengths of local cohomology modules with respect to various thickenings. Additionally, I study the surjectivity of local cohomology modules of thickenings of squarefree monomial ideals in low dimensions. The foundations I have already built while researching my thesis mean that I am particularly well-equipped to provide much needed concrete descriptions for existing work, as well as bringing to light new avenues to examine local cohomology modules. Thus the projects that follow fits in well with the current research directions of commutative algebra, while still providing a unique and unexplored interpretation.

2. Past Work

Problem 1. Explicitly construct an isomorphism between local cohomology modules.

Despite the abstract definition of local cohomology modules, I've demonstrated explicitly the isomorphism promised by the results of Bhatt, Blickle, Lyubeznik, Singh, and Zhang in the case of the ideal generated by size two minors of a 2×3 matrix.

We can define local cohomology modules with support in an ideal to be the cohomology of a Čech complex associated to that ideal. As an example, the Čech complex on R with support in the ideal generated by two elements, x and y, is:

$$0 \to R \to R_x \oplus R_y \to R_{xy} \to 0$$

I consider the case of $R = \mathbb{C}[X]$, where X is a 2×3 matrix of indeterminants and I the ideal generated by size two minors. Constructing a Čech complex with the six indeterminants that generate the homogenous maximal ideal, $\mathfrak{m}$, the local cohomology module $H^3_{\mathfrak{m}}(R/I^t)$ amounts to a homomorphic image of

$$\bigoplus_{i<j<k} \left(R_{x_i x_j x_k} \otimes R/I^t\right).$$

Therefore, $H^3_{\mathfrak{m}}(R/I^t)$ involves a direct sum of 20 different modules.

Surprisingly, the isomorphism $H^3_{\mathfrak{m}}(R/I^{t+1}) \to H^3_{\mathfrak{m}}(R/I^t)$ can be elegantly understood in terms of truncations of the formal power series of natural log.

Theorem 1. *[Ken18] Let $R = \mathbb{C}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$, i.e. $\mathbb{C}$ adjoin a 2×3 matrix of indeterminants, and let I be the ideal generated by size two minors. Then the identity*

$$\ln\left(\frac{wy}{vz}\frac{uz}{wx}\frac{vx}{uy}\right) = \ln(1) = 0$$

yields the Taylor series in generators of I:

$$\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\Delta_1}{vz}\right)^m + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\Delta_2}{wx}\right)^m + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\Delta_1}{uy}\right)^m = 0$$

The t^{th} truncation of this Taylor series can be used to construct the generator for $H^3_{\mathfrak{m}}(R/I^t)_0$.

Similarly, I also constructed a Taylor series formulation when $R = \mathbb{C}[u,v,x,y]$ and $I = (u^d + v^d, u^{d-1}x + v^{d-1}y, \ldots, x^d + y^d, uy - vx)$ for all integers $d \geq 2$. Note: the ring R/I is isomorphic to a Segre product of a hypersurface and a polynomial ring, $\mathbb{C}[a,b]/(a^d + b^d) \# \mathbb{C}[s,t]$. The generators of $H^1_{\mathfrak{m}}(R/I^t)_0$ rely on the Taylor series of $\frac{1}{\sqrt[d]{1-x}}$.

Bhatt et al.'s results depend on the characteristic of the ground field being zero; however, I have also considered the behavior of local cohomology modules in characteristic p [BBL$^+$16]. Here, the behavior of these modules is remarkably different and more complicated. I explicitly calculated elements of local cohomology modules in the characteristic p setting for the ring and ideal of theorem 1. Computer aided data suggests I have found all the elements. The size of these local cohomology modules indicates that an eventual isomorphism induced by thickenings is impossible, which is indeed supported by theoretical work.

Problem 2. Calculate the numerical invariant, $\epsilon^k(I)$, below, for a non-monomial ideal.

In a fixed graded component, Bhatt et al. showed isomorphisms of local cohomology modules exist [BBL$^+$16]. To consider every graded component is a larger problem. The numerical invariant

$$\epsilon^k(I) := \lim_{t \to \infty} \frac{\lambda\left(H^k_{\mathfrak{m}}(R/I^t)\right)}{t^{\dim(R)}}$$

measures the behavior of local cohomology modules in every degree. This invariant was first introduced in [KV10] as a generalization of multiplicity, $e(I)$, for which $\epsilon^0(I)$ agrees (up to a factor of d!) when I is $\mathfrak{m}$-primary. They showed deep connections between $\epsilon^0(I)$ and the analytic spread of I, as well as showing $\epsilon^0(I)$ can be used to detect integral closure. Cutkosky [CHST05] showed, for a broad group of rings, the limit $\epsilon^0(I)$ exists and demonstrated a case in which this invariant is not rational. This work was carried on by [HPV08], which showed $\epsilon^0(I)$ is rational when I is a monomial ideal. [DM17] then related ϵ^k to depth conditions on the Rees algebra, and calculated $\epsilon^k(I)$ for a monomial ideal.

Let $R = \mathbb{C}[X]$, where X is a $2 \times m$ matrix of indeterminates, and let I be the ideal generated by size two minors. By [Wit11], there are only two nonzero local cohomology modules of R/I^t; one at cohomological index $k = 3$ and one at $k = \dim(R/I) = m + 1$. The latter is infinite length and thus $\epsilon^{m+1}(I)$ is infinite for all m. Regarding the former, I describe $\epsilon^3(I)$ with the following.

Theorem 2. *[Ken18] Let $R = \mathbb{C}[X]$, where X is a $2 \times m$ matrix of indeterminates, and let I be the ideal generated by size two minors. Then we have the equality*

$$\epsilon^3(I) = \frac{1}{(m+1)(m!)^2}.$$

Problem 3. For which ideals, I is the natural map

$$H^k_{\mathfrak{m}}(R/I^{t+1}) \to H^k_{\mathfrak{m}}(R/I^t)$$

a surjection?

This is equivalent to a question of Eisenbud, Mustaţă, and Stillman [EMS00]. I have explored this question for squarefree monomial ideals in low dimensions. In the early 1970s, Stanley, Reisner, and Hochster identified quotients of polynomial rings by squarefree monomial ideals with finite simplicial complexes. This identification has allowed algebraists to examine quotients of rings by squarefree monomial ideals using both algebraic and geometric methods. This identification is helpful in many scenarios. However, I constructed the following example.

Example 3. *[Ken18] There exist two squarefree monomial ideals whose associated simplicial complexes are homotopy equivalent, yet the quotient rings give opposing answers to the question of Eisenbud, Mustaţă, and Stillman.*

While these two examples do not give a complete characterization of ideals satisfying the property in Problem 3, they do mean that local cohomology modules must detect some subtle structural information about rings that the simplicial complexes cannot. These examples enlighten us about the general nature of local cohomology as a robust theory to extract algebraic information about a ring.

3. Proposed Future Work

Problem 4. Determine the relationship between the Taylor series used to construct isomorphisms of local cohomology modules and the underying ring and ideal structures.

While I have computed the cohomology of the formal completion along a determinantal subvariety in one example, much remains to be done; I suspect that calculating more examples will shed better light on the problem. The presence of natural log in a local cohomology calculation is both beautiful and bizarre. Viewing the local cohomology module of size two minors of a 2×3 matrix through the lens of natural log is very illuminating, but the reason why this ring should be connected to natural log, of all functions, is still in shadow. Another calculation of the cohomology of a formal completion involves the Taylor series of $\frac{1}{\sqrt[d]{1-x}}$. Do these functions reveal some inner structure of the ring, or are they merely one lens of many through which to view isomorphisms of local cohomology modules?

In order to address this question, it is crucial to have more concrete constructions. My work constructing the explicit isomorphisms between local cohomology modules leaves me uniquely qualified to construct isomorphisms for broader classes of rings. Macaulay2 software does not work with Čech complexes. However, we can also consider local cohomology modules as a limit of Koszul homology, Koszul homology being a module which Macaulay2 can calculate. In my investigations so far, I have used these limits of Koszul homology to aid in making conjectures about elements of local cohomology, which I then prove using analytic techniques. I will extend this process to more quotient rings.

There is a subtlety to choosing quotient rings that are complicated enough to yield data, but not so complicated as to be intractable. If R/I is a hypersurface, then the thickenings R/I^t are also hypersurfaces, and therefore Cohen-Macaulay. The Bhatt et al. theorem [BBL+16] would then only apply to local cohomology modules that are already zero. Rather, I have had success producing isomorphisms between local cohomology modules of Segre products of hypersurfaces and polynomial rings, that is, rings of the class:

$$R/I \cong \mathbb{C}[\underline{x}]/(f)\#\mathbb{C}[\underline{y}]$$

where $\underline{x}$ and $\underline{y}$ are variables, and R is a regular ring. Specifically, for the class $\mathbb{C}[a,b]/(a^d + b^d)\#\mathbb{C}[s,t]$, I already constructed isomorphisms. Additionally, I am investigating Segre products of higher dimensional rings. A fascinating question is whether there is a relationship between the polynomial, f, and the Taylor series used to generate an isomorphism between local cohomology modules of thickenings of R/I.

Problem 5. Extend the results of Dao and Montaño to characteristic p settings.

Dao and Montaño proved a result on the limit ϵ^k in characteristic p, but the hypotheses do not apply to the case of an ideal generated by all size two minors of a 2×3 matrix. In particular, Dao and Montaño require that $H_I^3(R)$ be non-zero, which is not the case for R the 2×3 matrix of variables and I generated by size two minors, in characteristic p. Note that the Taylor series for natural log has coefficients $\frac{1}{m}$, which will not exist in characteristic p whenever p divides m. Nonetheless, a constant multiple of the Taylor series truncation will exist in characteristic p. However, the constant multiple depends on the thickening parameter, t. Thus, I constructed elements of $H_{\mathfrak{m}}^3(R/I^t)_0$ in characteristic p and the maps between them, but they lack the coherence seen in characteristic 0. This hands-on understanding of characteristic p behavior provides a solid foundation with which I can approach the existence of ϵ^k, first in the case of determinantal rings. A calculation of the limit in characteristic p in the determinantal ring case would shed light on the behavior of local cohomology modules of thickenings in characteristic p in general.

Problem 6. Explain the signifigance of ϵ^k as related to the underying ring and ideal structures.

I will use the represenation theoretic techniques of Raicu et al. [Rai18], [RWW14] to calculate lengths of local cohomology modules of more thickenings, and thus calculate more ϵ^k. The numerical invariant, multiplicity, ϵ^0, has a geometric interpretation, but the meaning of ϵ^k is still unclear. I will apply the techniques of Raicu to the GL-invariant ideal that generates the twisted-cubic, a heavily investigated ring that is a set theoretic complete intersection but not an ideal theoretic complete intersection.

Under mild assumptions, the invariant Hilbert-Samuel multiplicity, which agrees with $\epsilon^0(\mathfrak{m})$, detects the regularity of the ring. Furthermore, Katz and Validashti showed $\epsilon^0(I)$ detects the analytic spread of the ideal, I, and Dao and Montaño extended this to showing the positivity of ϵ^k gives a lower bound on analytic spread. I will strengthen their results and explore more facets of ϵ^k.

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